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VARIATIONS ON THE THEME OF REPEATED DISTANCES

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We give an asymptotically sharp estimate for the error term of the maximum number of unit distances determined by n points in $\mathbf{R}^d$, $d \geq 4$. We also give asymptotically tight upper bounds on the total number of occurrences of the "favourite" distances from n points in $\mathbf{R}^d$, $d \geq 4$. Related results are proved for distances determined by n disjoint compact convex sets in $\mathbf{R}^2$.

1. Unit distances

Given a set $X = \{x_1, \dots, x_n\}$ of n points in $\mathbf{R}^d$, let $f(X)$ denote the number of pairs $\{x_i, x_j\}$ whose Euclidean distances $\|x_i - x_j\| = 1$. Let

$$f_d(n) = \max_{\substack{X \subset \mathbf{R}^d \\ |X| \leq n}} f(X).$$

In other words, $f_d(n)$ is the maximum number of unit distances determined by n points in d-dimensional Euclidean space. Clearly, $f_1(n) = n - 1$, but it seems to be very difficult to find even the right order of magnitude of $f_2(n)$ and $f_3(n)$. The best known bounds

$$n^{1+(c/\log\log n)} < f_2(n) < c' n^{4/3},$$
$$cn^{4/3} \log\log n < f_3(n) < n^{3/2}\beta(n)$$

are due to Erdős [8], Spencer – Szemerédi – Trotter [16], Erdős [9] and Clarkson – Edelsbrunner – Guibas – Sharir – Welzl [6], respectively. ($\beta(n)$ is closely related to the inverse of Ackermann's function, and grows extremely slowly.)

However, the asymptotic behaviour of $f_d(n)$ is fairly well known for $d \geq 4$. Erdős [9,10] showed that in this case

$$f_d(n) = \frac{n^2}{2}\left(1 - \frac{1}{\lfloor d/2 \rfloor}\right) + O(n^{2-\varepsilon_d})$$

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with some very small constant $\varepsilon_d > 0$. Moreover, he proved that, if $d \geq 4$ is *even*, then the error term is *linear* in n. In the same paper, written more than twenty years ago, Erdős complained that he was unable to decide whether

$$f_d(n) = \frac{n^2}{2}\left(1 - \frac{1}{\lfloor d/2 \rfloor}\right) + O(n)$$

holds for *every* fixed $d \geq 4$. Our next theorem answers this question in the negative.

Theorem 1. *Given any odd $d \geq 5$, there exist $c_d, c_d' > 0$ such that for every sufficiently large n*

$$\frac{n^2}{2}\left(1 - \frac{1}{\lfloor d/2 \rfloor}\right) + c_d n^{4/3} < f_d(n) < \frac{n^2}{2}\left(1 - \frac{1}{\lfloor d/2 \rfloor}\right) + c_d' n^{4/3}.$$

Proof. First we establish the lower bound. Let $d = 2r + 1$, and fix r mutually orthogonal subspaces $F_1, \ldots, F_r \subseteq \mathbb{R}^d$ (through the origin 0) such that $\dim F_1 = 3$, $\dim F_i = 2$ for all $2 \leq i \leq r$. Let $S_1 \subseteq F_1$ and $C_i \subseteq F_i$, $2 \leq i \leq r$ denote the sphere of radius $1/\sqrt{2}$ and the circle of radius $1/\sqrt{2}$ centered at 0, respectively.

One can choose a set V_1 of $m = \lfloor n/r \rfloor$ points on S_1 so that they determine at least $\gamma m^{4/3}$ unit distances. To see this, observe that $x, y \in S_1$ are at distance 1 if and only if the corresponding vectors are perpendicular to each other. Let $P \subseteq F_1$ be a 2-dimensional plane not containing 0. According to an old construction due to Erdős (see e.g. [7]), one can pick $\lfloor m/2 \rfloor$ points and $\lceil m/2 \rceil$ straight lines in P such that the total number of incidences between them is at least $\gamma m^{4/3}$ (for some absolute constant $\gamma > 0$). To each point p of this system we assign a vector $x_p \in F_1$ of length $1/\sqrt{2}$ pointing from 0 towards p. To each line ℓ of this system we assign a vector $y_\ell \in F_1$ of length $1/\sqrt{2}$ orthogonal to the plane spanned by ℓ and 0. Obviously, x_p is perpendicular to y_ℓ whenever $p \in \ell$. Thus, the points $x_p, y_\ell \in S_1$ determine at least $\gamma m^{4/3}$ unit distances.

For any other i ($2 \leq i \leq r$), let V_i denote an arbitrary $m = \lfloor n/r \rfloor$ element subset of C_i. Any two points belonging to different V_i's are at distance 1 from each other, hence the total number of unit distances determined by $X = V_1 \cup \ldots \cup V_r$,

$$f(x) \geq \binom{r}{2} m^2 + \gamma m^{4/3} \geq \frac{n^2}{2}\left(1 - \frac{1}{r}\right) + c_d n^{4/3}.$$

Next we show the upper bound. Let us fix an n-element subset $X \subseteq \mathbb{R}^{2r+1}$, and let G denote the graph whose vertex set is X, with two vertices $x, y \in X$ being joined by an edge whenever $\|x - y\| = 1$. It is well known (see [9]) and can readily be checked that $K_{r+1}(3) \not\subseteq G$, i.e., G does not contain a complete $(r+1)$-partite subgraph with 3 points in each of its classes. Thus, we can apply to G the following result of Bollobás – Erdős – Simonovits – Szemerédi [3] (see also [2]).

Lemma 2. *Let r be a fixed natural number, $\varepsilon > 0$. Then there exists a constant $c_{\varepsilon,r}$ with the property that the vertex set of every graph G with n vertices and at least $n^2(1 - 1/r)/2$ edges satisfying $K_{r+1}(3) \not\subseteq G$ can be partitioned into r almost equal classes*

$$V(G) = V_1 \cup \ldots \cup V_r, \quad \left| |V_i| - \frac{n}{r} \right| < \varepsilon n \quad (1 \leq i \leq r)$$

such that, with the exception of at most $c_{\varepsilon,r}$ points, every point of G has at most εn neighbours in its own class and is connected to all but at most εn points in the other classes.

Suppose that n is sufficiently large, $\varepsilon < 1/(10r)$, and apply the lemma to the unit distance graph G defined above. For any i, let V_i' denote the set of nonexceptional elements of V_i, i.e., the set of points connected to all but at most εn points of $\bigcup_{j\neq i} V_j$. We claim that V_i' is contained in some sphere S_i of a 3-dimensional flat $F_i \subseteq \mathbf{R}^{2r+1}$, $1 \leq i \leq r$.

It suffices to prove that any 5 points of V_i', say, are *cospherical*, i.e., lie on the boundary of the same 3-dimensional ball. Pick any $x_{11}, \ldots, x_{15} \in V_1'$. We can choose recursively $x_{jk} \in V_j'$ $(2 \leq j \leq r,\ 1 \leq k \leq 3)$ such that $\|x_{jk} - x_{i\ell}\| = 1$ whenever $i \neq j$. Let C_j denote the circle determined by x_{j1}, x_{j2}, x_{j3} $(2 \leq j \leq r)$. Evidently, all of these circles must have the same centre c, and span mutually orthogonal planes $F_2, \ldots, F_r$. This, in turn, implies that $x_{11}, \ldots, x_{15}$ are contained in the 3-dimensional flat F_1 passing through C and orthogonal to all F_j $(2 \leq j \leq r)$. In view of the fact that $x_{11}, \ldots, x_{15}$ are equidistant from x_{21}, we obtain that they are cospherical, as required.

On the other hand, let $f_3^*(m)$ denote the number of times the same distance can occur among m cospherical points. It follows by the methods of a recent paper of Clarkson – Edelsbrunner – Guibas – Sharir – Welzl ([6], cf. also [11]) that there exists a constant δ such that $f_3^*(m) \leq \delta m^{4/3}$ for any m. Thus

$$\begin{aligned} f(X) &\leq \sum_{1\leq i<j\leq r} |V_i'||V_j'| + \sum_{1\leq i\leq r} f_3^*(|V_i'|) + O(n) \\ &\leq \frac{n^2}{2}\left(1 - \frac{1}{r}\right) + r\delta n^{4/3} + O(n), \end{aligned}$$

which proves the upper bound in Theorem 1. ∎

Remark. With a little extra care, the above argument also yields that, if $\lim_{n\to\infty} f_3^*(n)/n^{4/3}$ exists, then so does

$$\lim_{n\to\infty} \frac{f_d(n) - \frac{n^2}{2}\left(1 - \frac{1}{\lfloor d/2 \rfloor}\right)}{n^{4/3}} > 0$$

for every odd $d \geq 5$.

2. Favourite distances

In [1] we investigated the following problem. Let $X = \{x_1, \ldots, x_n\}$ be a set of n points in $\mathbf{R}^d$, and let r_i denote the radius of a sphere around x_i containing the maximum number of elements of X. In what follows, r_i is called a *favourite distance* from x_i, and t_i denotes the number of occurrences of this distance, i.e.,

$$t_i = |\{x_j : \|x_j - x_i\| = r_i\}|, \quad 1 \leq i \leq n.$$

We want to estimate

$$F_d(n) = \max_{\substack{X\subset\mathbb{R}^d \\ |X|\le n}} \sum_{i=1}^{n} t_i.$$

Using the notation of the previous section, we obviously have that $F_d(n) \geq 2f_d(n)$ for all d and n. In particular, for any fixed $d \geq 2$,

$$F_d(n) \geq n^2 \left(1 - \frac{1}{\lfloor d/2 \rfloor} + o(1)\right).$$

In [1] we showed that this bound is asymptotically tight in any *even* dimensional space. Here we are going to present a proof that works for all d.

Theorem 3. *Given any* $d \geq 4$,

$$F_d(n) = n^2 \left(1 - \frac{1}{\lfloor d/2 \rfloor} + o(1)\right),$$

while n tends to infinity.

Proof. We have to establish only the upper bound. Let $X = \{x_1, \ldots, x_n\} \subseteq \mathbb{R}^d$, and let r_i denote a favourite distance from x_i. Define a directed graph G on the vertex set $V(G) = X$ such that there is a directed edge from x_i to x_j,

$$\overrightarrow{x_i x_j} \in E(G) \Longleftrightarrow \|x_i - x_j\| = r_i.$$

Note that it can happen that both $\overrightarrow{x_i x_j}$ and $\overrightarrow{x_j x_i}$ are edges of $\overrightarrow{G}$. Clearly,

$$|E(\overrightarrow{G})| = \sum_{i=1}^{n} t_i.$$

Given any directed graph $\overrightarrow{K}$ and any natural number s, let $\overrightarrow{K}(s)$ denote the directed graph obtained by replacing each vertex of $\overrightarrow{K}$ with a class of s independent points, and joining two points in different classes by an edge if and only if the corresponding vertices of $\overrightarrow{K}$ are connected in the same way.

Lemma 4. *Let* $\overrightarrow{K}_{\lfloor d/2 \rfloor+1}$ *be a tournament with* $\lfloor d/2 \rfloor + 1$ *vertices, and assume that the indegree of each vertex is at least 1. Then* $\overrightarrow{K}_{\lfloor d/2 \rfloor+1}(3) \not\subseteq \overrightarrow{G}$.

Proof. Suppose this is false. Then each class of $\overrightarrow{K}_{\lfloor d/2 \rfloor+1}(3)$ consists of 3 noncollinear points, and the planes spanned by them must be mutually orthogonal. This would imply that the dimension of the whole space is at least $2(\lfloor d/2 \rfloor + 1) \geq d + 1$, a contradiction. ∎

Given any directed graph $\overrightarrow{K}$, let $\mathrm{ex}(n, \overrightarrow{K})$ denote the maximum number of edges of a directed graph with n vertices containing no subgraph isomorphic to $\overrightarrow{K}$. The following result was proved by Brown – Harary [4], who initiated the investigation of Turán–type extremal questions for directed graphs.

Lemma 5. *For any tournament $\overrightarrow{K}_{p+1}$ with $p+1$ vertices, and for any natural number n,*

$$\mathrm{ex}(n, \overrightarrow{K}_{p+1}) = n^2\left(1-\frac{1}{p}\right) - \frac{q(p-q)}{p},$$

where q denotes the remainder of n upon division by p. ∎

To see that $\mathrm{ex}(n, \overrightarrow{K}_{p+1})$ is *at least* as large as the expression on the right-hand side, take a complete (undirected) p-partite graph of n vertices, whose classes are as equal as possible, and replace each of its edges by two oriented edges, one in each direction.

Finally, we need a result of Brown – Simonovits [5], that can be regarded as a directed version of the Erdős – Stone theorem [12].

Lemma 6. *Given any directed graph $\overrightarrow{K}$ and any natural number s,*

$$\mathrm{ex}(n, \overrightarrow{K}(s)) = \mathrm{ex}(n, \overrightarrow{K}) + o(n^2).$$ ∎

Combining the lemmata above, we obtain that if $\overrightarrow{K}_{\lfloor d/2\rfloor+1}$ is a tournament whose no vertex has indegree 0, then

$$\begin{aligned} F_d(n) = \max |E(\overrightarrow{G})| &\le \mathrm{ex}(n, \overrightarrow{K}_{\lfloor d/2\rfloor+1}(3)) \\ &\le \mathrm{ex}(n, \overrightarrow{K}_{\lfloor d/2\rfloor+1}) + o(n^2) = n^2\left(1-\frac{1}{\lfloor d/2\rfloor}+o(1)\right). \end{aligned}$$ ∎∎

For $d=2$ and 3, the best currently known upper bounds were established in [6]: $F_2(n) = O\left(n^{7/5}\right)$, $F_3(n) = O\left(n^{13/7}\beta(n)\right)$.

3. Distances between convex sets

The *distance* of two sets $C, D \subseteq \mathbf{R}^d$ is defined by

$$\delta(C,D) = \inf_{\substack{c\in C\\ d\in D}} \|c-d\|.$$

Given a family of pairwise *disjoint compact convex* sets $C_1,\dots,C_n \subseteq \mathbf{R}^d$, let $h(C_1,\dots,C_n)$ denote the number of pairs $\{C_i, C_j\}$ whose distance $\delta(C_i,C_j)=1$. Let

$$h_d(n) = \max_{C_1,\dots,C_n\subseteq \mathbf{R}^d} h(C_1,\dots,C_n),$$

i.e., the maximum number of unit distances determined by n disjoint compact convex sets in Euclidean d-space. The function $h_d^T(n)$ is defined simlarly, except that now the maximum is taken over all families $\{C_1,\dots,C_n\}$ whose members are *translates* of each other.

With the notation of Section 1, we obviously have that

$$h_d(n) \geq h_d^T(n) \geq f_d(n) \quad (\forall d\, \forall n).$$

However, we conjecture that already in the plane ($d = 2$) $h_d^T(n)$ can be much larger than $f_d(n)$. In particular, we propose the following

Conjecture. There exists an absolute constant $\varepsilon > 0$ such that $h_2^T(n) > n^{1+\varepsilon}$ for every sufficiently large n.

Theorem 7. *(i)* $h_2(n) = O(n^{7/5})$,
(ii) $h_2^T(n) = O(n^{4/3})$,

Proof. Let $\{C_1, \dots, C_n\}$ be a system of pairwise disjoint compact convex sets in the plane. Denote by G the undirected graph on the vertex set $C(G) = \{C_1, \dots, C_n\}$, where C_i and C_j are connected by an edge whenever $\delta(C_i, C_j) = 1$.

Let D denote the closed unit disk, and let $+$ stand for the Minkowski sum. Clearly, $\delta(C_i, C_j) = 1$ if and only if $C_i + \frac{1}{2}D$ and $C_j + \frac{1}{2}D$ touch each other. We shall make use of the following assertion from [13].

Lemma 8. *For any two integers $i \neq j$, the convex curves bounding $C_i + \frac{1}{2}D$ and $C_j + \frac{1}{2}D$ have at most 2 points in common.* ∎

Lemma 9. *G has no subgraph isomorphic to $K_2(3)$ — a complete bipartite graph with 3 vertices in each of its classes.*

Proof. Assume that the lemma is false, and, say, C_1, C_2, C_3 and C_4, C_5, C_6 are the two classes of a $K_2(3)$. Put $C_i' = C_i + \frac{1}{2}D$. Then $C_1' \cup C_2' \cup C_3'$ and $C_4' \cup C_5' \cup C_6'$ have no interior points in common. By Lemma 8, $\mathbb{R}^2 - (C_1' \cup C_2' \cup C_3')$ has at most 2 connected components, and it is not difficult to see that at least one of them must contain 2 sets C_i' and C_j' (for some $4 \leq i < j \leq 6$) having more than 2 boundary points in common, a contradiction. The details are left to the reader. ∎

Similarly, we can easily establish

Lemma 10. *If $C_1, \dots, C_n$ are translates of the same compact convex set, then G does not have a complete bipartite subgraph $K_2(2,3)$ with 2 vertices in one of its classes and 3 in the other one.*

Thus, the following well-known result of Kővári – Sós – Turán [14] and Erdős can be applied.

Lemma 11. *Let $s \geq r$ be fixed natural numbers. Then there exists a constant c_r such that any graph with n vertices, containing no subgraph isomorphic to $K_2(r,s)$, has at most $c_r n^{2-(1/r)}$ edges.* ∎

Hence, we obtain (i) $h_2(n) = O(n^{5/3})$, (ii) $h_2^T(n) = O(n^{3/2})$. Using these bounds, the random sampling technique developed in [6] established the sharper estimates in the theorem. ∎∎

4. Open problems

In this field it is very easy to raise questions which are extremely difficult to answer. We close this paper by proposing a couple of problems that are perhaps not completely hopeless.

Problem 1. Given a set $X = \{x_1, \dots, x_n\}$ of n points in $\mathbb{R}^d$, let $t(X)$ denote the number of distinct distances determined by X. Let $t_d(n) = \min t(X)$, where the minimum is taken over all n-element point sets X in $\mathbb{R}^d$ which do not contain an isosceles triangle, i.e., $\|x_i - x_j\| \neq \|x_i - x_h\|$ $(\forall i \neq j \neq h)$. Is it true that
(i) $\lim_{n\to\infty} t_d(n)/n = \infty$ for every fixed dimension d?
(ii) $\lim_{n\to\infty} t_{d+1}(n)/t_d(n) = 0$ for every fixed d?

Problem 2. Let $x_1, \dots, x_n$ be n distinct points in the plane in general position (no 3 on a line, no 4 on a circle). Denote by $t(x_i)$ the number of distinct distances from x_i. Clearly, $t(x_i) \geq (n-1)/3$ for every i. Does there exist a positive ε such that

(i) $$\max_{1\leq i\leq n} t(x_i) \geq \left(\frac{1}{3} + \varepsilon\right) n,$$

(ii) $$\sum_{1\leq i\leq n} t(x_i) \geq \left(\frac{1}{3} + \varepsilon\right) n^2,$$

provided that n is sufficiently large?

Assume further that any circle around a point x_i contains at most 2 other x_j's. Is it then true that

(iii) $$\max_{1\leq i\leq n} t(x_i) = (1 - o(1))\, n,$$

or at least

(iv) $$\max_{1\leq i\leq n} t(x_i) = \left(\frac{1}{2} + \varepsilon\right),$$

for some $\varepsilon > 0$?

Problem 3. Let $T_d(n)$ denote the minimum number of distinct distances determined by n points in $\mathbb{R}^d$. Given two n-element point sets $X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_n\}$ in $\mathbb{R}^d$, let $T(X, Y)$ denote the number of distinct distances $\|x_i - y_j\|$, and put $T_d^{\text{bip}}(n) = \min_{X,Y} T(X, Y)$. (Here "bip" stands for bipartite.) Prove or disprove that

$$\lim_{n\to\infty} \frac{T_d^{\text{bip}}(n)}{T_d(n)} = 0$$

for $d = 2$ or 3. For $d \geq 4$ this obviously follows from the construction of Lenz.

Problem 4. Let $x_1, \dots, x_n$ be n distinct poins in the plane, and let $d_{\min} = d_1 < d_2 < \dots < d_k = d_{\max}$ denote the distinct distances determined by them. Assume that d_i occurs s_i times. Prove or disprove

(i) $$s_{\min} s_{\max} \leq \frac{9}{8} n^2 + o(n^2).$$

This inequality, if true, cannot be improved, as is shown by an easy construction of E. Makai, Jr. It is not difficult to prove that

(ii) $$s_{\min} + s_{\max} \leq 3n - c\sqrt{n} + o(\sqrt{n})$$

for some $c > 0$. Determine the best possible value of the constant c. Is it true that

(iii) $$s_{\min} \leq 3n - 2m + o(\sqrt{n}),$$

where m denotes the number of vertices of the convex hull of $\{x_1, \ldots, x_n\}$? Note that (i) can readily be deduced from (iii).

The odd regular polygon shows that it is feasible that every s_i is at least n. Can it happen that

(iv) $$\min_{1 \leq i < k} s_i \geq n + 1$$

for some choice of the x_i? It is well-known that $s_{\max} \leq m \leq n$.

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